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ELASTO–PLASTIC CONSOLIDATION OF SOIL

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Abstract—A theory of the behaviour of an ideal mass of two-phase soil is developed. This theory unites the usually separately considered aspects of analytical soil mechanics of settlement and deformation; time dependent consolidation; and yielding leading to collapse of the mass. The theory assumes the soil to be an elasto–plastic permeable material and a finite element approximation of the resulting equations is derived through the principle of virtual work. Two example problems are treated numerically.

1. INTRODUCTION

The analysis of a foundation resting on a saturated clay is an important problem in foundation engineering. The analysis is usually divided into two distinct parts, an investigation of stability and an investigation of the settlement of the foundation.

Saturated clay is a two phase material consisting of a compressible solid phase (the skeleton of soil particles) and a liquid phase (the water filling the pores of the skeleton). It is this two-phase nature which causes saturated clay to exhibit two different extreme behaviours: the behaviour under very slow loading or a very long time after load application known as drained conditions, and the behaviour under very rapid loading known as undrained conditions. The basis of theoretical analysis is Terzaghi's effective stress concept which assumes that the total normal stress on any plane is the algebraic sum of the normal stress in the soil skeleton (the effective normal stress) and the pore water pressure. When a load is applied to a saturated clay there is a change in total stress, this leads to a change in effective stress and to the development of excess pore pressures (increase in pore pressure above its initial value). If the load is applied so quickly, that these excess pore pressures cannot dissipate, the clay is said to behave in an undrained manner, which is characterised by undrained strength and deformation properties. If however the loads are applied so slowly that no excess pore pressures develop, then the clay is said to behave in a drained manner. The drained behaviour is governed by drained or effective stress strength and deformation parameters. The undrained and drained behaviour of clay represent two extremes of behaviour. In general when a clay is first loaded it responds in an undrained manner and undergoes an initial settlement, the excess pore pressures then start to dissipate and this leads to a characteristic time-settlement behaviour and the clay is said to consolidate. After a long period of time when all excess pore pressures have dissipated the clay behaves in a drained manner.

The stability analysis usually consists of an investigation of the short term stability of the foundation using undrained parameters, and an investigation of its long term stability using drained or effective stress parameters. Each of these stability analyses may be performed in a variety of ways. One approach would be to use an approximate engineering analysis such as the "slip circle". A more rigorous approach would be to use the classical theory of plasticity [1–9]. For many practical problems it is not possible to find an exact collapse load and so it is necessary, either to use the limit theorems [1, 2] to find sufficiently close upper and lower bounds (provided the material has an associated flow rule), or to adopt a loading path technique [10–14].

The settlement analysis usually consists of an investigation of the initial settlement using undrained parameters, and an investigation of the final settlement using drained parameters, together with an investigation of the rate at which settlement occurs.

The initial and final settlements are usually obtained by using elastic theory and the appropriate elastic constants, although it would be possible to use elasto–plastic loading path techniques, while a good approximation to the rate of settlement can be found using three dimensional diffusion theory. A comprehensive treatment of this approach is given in Ref. [15].

A more rigorous treatment of the time settlement behaviour can be obtained by using Biot's consolidation theory [16–19]. In practical problems it is usually necessary to integrate these equations numerically, and numerical approaches, for the case when the soil skeleton is elastic, have been developed by various authors[20–24].

The division of the foundation analysis into a stability investigation and a settlement investigation is somewhat artificial. Traditional stability analyses can give no indication of the safety factor when loading proceeds at a rate too slow for the soil to be considered undrained yet too fast for the soil to be considered drained. In just the same way traditional settlement and rate of settlement analyses, based on the assumption of an elastic soil skeleton, may well be in error in situations in which the loading induces significant local yielding.

The necessity for such a division can be removed by developing the equations governing the consolidation of a soil with an elasto–plastic skeleton. This allows for interplay between the processes of yielding and consolidation and provides the means for analysing the behaviour of the foundation under any given loading path.

2. BIOT'S EQUATIONS FOR A SOIL WITH AN ELASTIC SKELETION

Biot's equations governing the consolidation of a saturated elastic solid under quasi-static conditions may be derived from the following considerations. (a) The stresses are in equilibrium. (b) The effective stresses are related to the strains by Hooke's Law. (c) The flow of water through the soil is governed by Darcy's Law. (d) The pore water is incompressible when compared to the soil skeleton and thus rate of volume decrease of the element equals the rate at which water is expelled. This leads to the following equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} - F_i = 0 \tag{1a}$$

$$\sigma'_{ij} = \sigma_{ij} - p\delta_{ij} = -H_{ijkl}\epsilon_{kl} \tag{1b}$$

$$v_i = -\frac{k_{ij}}{\gamma_w}\frac{\partial p}{\partial x_j} \tag{1c}$$

$$\frac{\partial v_i}{\partial x_i} = -\frac{\partial \theta}{\partial t} \tag{1d}$$

where

- x_i are the coordinates in some Cartesian reference system
- σ_{ij} are the components of the total stress tensor. Compression considered as positive
- F_i are the components of body force
- p the pore pressure
- σ'_{ij} are the components of effective stress $\sigma'_{ij} = \sigma_{ij} - p\delta_{ij}$
- u_i are the components of the displacement vector
- v_i are the components of the superficial velocity vector. (The average true velocity is v_i/porosity.)
- ϵ_{ij} are the components of the strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- θ is the volume strain ($\theta = \epsilon_{ii}$)
- H_{ijkl} are the elastic coefficients in the generalized Hooke's law
- k_{ij} are the coefficients of permeability in the generalized Darcy's law
- γ_w is the unit weight of water
- t denotes the time.

It should be noted at this stage that the quantities (σ_{ij}, F_i, p, σ'_{ij}, u_i, v_i) all represent excesses over the initial values of (σ_{ijo}, F_{io}, p_o, σ'_{ijo}, u_{io}, v_{io}) and thus, for example, current stress σ_{ijc} would be given by

$$\sigma_{ijc} = \sigma_{ijo} + \sigma_{ij}.$$

In order to be specific the soil is assumed to occupy a region V, part of its surface S_T is subjected to applied tractions T_i, while the remainder of its surface S_D is subjected to zero displacements. The surface may also be divided into a portion S_p which is free to drain, while the remainder S_I is assumed to be impermeable† (see Fig. 1).

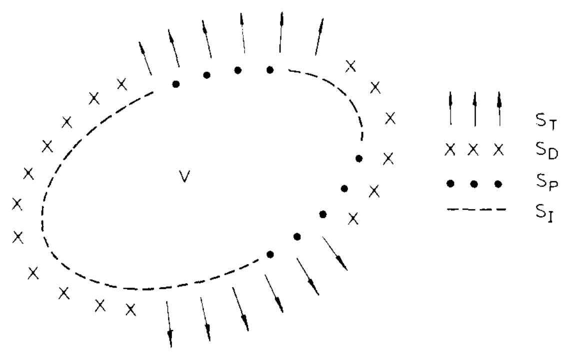

Fig. 1.

For this problem the boundary conditions are:

$$\sigma_{ij}n_j = -T_i \quad \text{on} \quad S_T \tag{2a}$$

$$u_i = 0 \quad \text{on} \quad S_D \tag{2b}$$

$$p = 0 \quad \text{on} \quad S_P \tag{2c}$$

$$n_i v_i = 0 \quad \text{on} \quad S_I \tag{2d}$$

where n_i denotes the outward normal to S.

Equations (1a–d) must be integrated subject to the boundary conditions (2a–d) and the initial condition:

$$\theta = 0 \quad \text{when} \quad t = 0^+. \tag{2e}$$

This last equation follows from the assumption that the pore water is incompressible relative to the soil skeleton. Thus there can be no instantaneous volume change even though a load is applied suddenly.

3. FINITE ELEMENT EQUATIONS FOR A SOIL WITH AN ELASTIC SKELETON

The finite element equations for a consolidating elastic soil have been derived in a variety of ways, for example Sandhu and Wilson[20] have used the Gurtin type of variational principal, while the authors[23] have used a variational theorem involving the Laplace transforms of the field quantities. In this paper it proves most convenient to adopt the following simplified approach.

Let u_i and $u_i + du_i$ be displacement fields which satisfy the boundary conditions on S_D. Then it is well known that the equilibrium eqns (1a) and the stress boundary conditions (2a) are satisfied if and only if

$$\int \{\sigma_{ij}d\epsilon_{ij} + F_i du_i\}\, dV + \int T_i du_i dS = 0. \tag{3a}$$

On inserting Hooke's law this equation becomes:

$$\int \{-H_{ijkl}\epsilon_{kl}d\epsilon_{ij} + p\,d\theta\}\, dV + \int F_i du_i dV + \int T_i du_i dS = 0. \tag{3b}$$

†The extension to more complicated boundary conditions, both elastic and hydraulic, is straight-forward and will not be given here.

Similarly let p, $p + dp$ be two sets of pore pressures which satisfy the boundary conditions on S_p, then eqns (1d, 2d) are satisfied if and only if:

$$\int \left\{ v_i \frac{\partial dp}{\partial x_i} - \frac{\partial \theta}{\partial t} dp \right\} dV = 0. \tag{3c}$$

On inserting Darcy's law this becomes:

$$-\int \left\{ k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial dp}{\partial x_i} + \frac{\partial \theta}{\partial t} dp \right\} dV = 0. \tag{3d}$$

An approximate solution of eqns (3b, d) may be obtained by an application of the finite element technique.

(a) Suppose that $\mathbf{u}$, p can be adequately represented by their values at the nodes 1, 2, 3, . . .

$$\boldsymbol{\delta}^T = (\mathbf{u}_1{}^T, \mathbf{u}_2{}^T, \ldots) \tag{4a}$$

$$\mathbf{q}^T = (p_1, p_2, \ldots). \tag{4b}$$

(b) Suppose also that the continuous values of $\mathbf{u}$, p may be approximated in terms of their nodal values

$$\mathbf{u} = \mathbf{C}(x_i)\boldsymbol{\delta} \tag{4c}$$

$$p = \mathbf{a}^T(x_i)\mathbf{q} \tag{4d}$$

where the form of $\mathbf{C}$, $\mathbf{a}$ will depend upon the particular type of finite element adopted.

(c) Equations (4c, d) may now be used to obtain the following approximations:

$$\boldsymbol{\epsilon} = \mathbf{B}(x_i)\boldsymbol{\delta} \tag{4e}$$

$$\theta = \mathbf{d}^T(x_i)\boldsymbol{\delta} \tag{4f}$$

$$\nabla p = \mathbf{E}(x_i)\mathbf{q} \tag{4g}$$

where

$$\mathbf{B} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ 0 & \partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \end{bmatrix} \mathbf{C}(x_i)$$

$$\mathbf{d}^T = (1, 1, 1, 0, 0, 0)\mathbf{B}$$

$$\mathbf{E} = \begin{bmatrix} \dfrac{\partial \mathbf{a}^T}{\partial x_1} \\[2ex] \dfrac{\partial \mathbf{a}^T}{\partial x_2} \\[2ex] \dfrac{\partial \mathbf{a}^T}{\partial x_3} \end{bmatrix}$$

and where $\boldsymbol{\epsilon} = (\epsilon_{11}, \epsilon_{22}, \ldots)^T$ is the vector of strain components.

If eqns (4c–f) are substituted into eqn (3b) then it is found that

$$d\boldsymbol{\delta}^T \{\mathbf{K}_E\boldsymbol{\delta} - \mathbf{L}^T\mathbf{q} - \mathbf{m}\} = \mathbf{0} \tag{5a}$$

where

$$\mathbf{K}_E = \int \mathbf{B}^T\mathbf{D}_E\mathbf{B}\, dV$$

$$\mathbf{L}^T = \int \mathbf{da}^T\, dV$$

$$\mathbf{m} = \int \mathbf{C}^T\, \mathbf{T}\, dS + \int \mathbf{C}^T\mathbf{F}\, dV$$

and where as usual $\mathbf{D}_E$ denotes the matrix of elastic constants for the stress strain law $\boldsymbol{\sigma} = \mathbf{D}_E\boldsymbol{\epsilon}$.

Equation (5a) is true for arbitrary variation $d\boldsymbol{\delta}$ and thus

$$\mathbf{K}_E\boldsymbol{\delta} - \mathbf{L}^T\mathbf{q} = \mathbf{m}. \tag{5b}$$

In similar fashion if eqns (4d, f, g) are substituted into eqn (3d) then it is found that

$$d\mathbf{q}^T\left\{-\mathbf{L}\frac{d\boldsymbol{\delta}}{dt} - \boldsymbol{\Phi}\mathbf{q}\right\} = \mathbf{0} \tag{5c}$$

where

$$\boldsymbol{\Phi} = \frac{1}{\gamma_W}\int (\mathbf{E}^T\mathbf{k}\mathbf{E})\, \mathrm{d}V$$

$\mathbf{k}$ is the matrix of permeability coefficients for Darcy's Law

$$\mathbf{v} = -\mathbf{k}\nabla p$$

and where $\mathbf{L}$ was defined previously.

Equation (5c) is true for arbitrary variations $d\mathbf{q}$ and thus

$$-\mathbf{L}\frac{d\boldsymbol{\delta}}{dt} - \boldsymbol{\Phi}\mathbf{q} = \mathbf{0}. \tag{5d}$$

Equations (5b, d) may be integrated once initial values of $\mathbf{q}$, $\boldsymbol{\delta}$ are known. These may be obtained either by performing a separate elastic analysis using undrained elastic parameters or by adopting the artifice that the loads increase from an initial value of zero over a short period of time.

4. FINITE ELEMENT EQUATIONS FOR A GENERAL INCREMENTAL LAW

Suppose that the material under consideration is in a given state (δ_{ij}, u_i, p, etc) due to the applied tractions T_i, and the body forces F_i. Now suppose these tractions and body forces increase at the rate $\dot{T}_i$, $\dot{F}_i$ respectively, where the dot denotes differentiation with respect to time. The question now arises as to what will be the corresponding rate of increase ($\dot{\sigma}_{ij}$, $\dot{u}_i$, $\dot{p}\dots$) of (stress, displacement, pore pressure etc.). A simple answer to this question may be obtained by merely differentiating eqns (5b, d), however it is more instructive to proceed as follows.

Differentiate eqns (1a–d) with respect to time then:

$$\frac{\partial \dot{\sigma}_{ij}}{\partial x_j} - \dot{F}_i = 0 \tag{6a}$$

$$\dot{\sigma}'_{ij} = \dot{\sigma}_{ij} - \dot{p}\delta_{ij} = -H_{ijkl}\dot{\epsilon}_{kl} \tag{6b}$$

$$\dot{v}_i = -\frac{k_{ij}}{\gamma_w}\frac{\partial \dot{p}}{\partial x_j} \tag{6c}$$

$$\frac{\partial \dot{v}_i}{\partial x_i} = -\frac{\partial \dot{\theta}}{\partial t} \tag{6d}$$

Differentiate the boundary conditions (2a–d) with respect to time then

$$\dot{\sigma}_{ij}n_j = -\dot{T}_i \quad \text{on} \quad S_T \tag{7a}$$

$$\dot{u}_i = 0 \quad \text{on} \quad S_D \tag{7b}$$

$$\dot{p} = 0 \quad \text{on} \quad S_p \tag{7c}$$

$$n_i\dot{v}_i = 0 \quad \text{on} \quad S_I. \tag{7d}$$

It is now clear that there is a direct analogy between the quantities (σ_{ij}, u_i, $p, \ldots T_i$, F_i) in eqns (1a–d) and (2a–d) and the quantities ($\dot{\sigma}_{ij}$, $\dot{u}_i$, $\dot{p}, \ldots \dot{T}_i$, $\dot{F}_i$) in eqns (6a–d) and (7a–d) and thus the arguments of Section 3 may be used to derive finite element equations for $\dot{\boldsymbol{\delta}}$ and $\dot{\mathbf{q}}$. On repeating these arguments it will be noted that they remain valid for any incremental stress strain law of the form:

$$\dot{\boldsymbol{\sigma}} = -\mathbf{D}_I\dot{\boldsymbol{\epsilon}} \tag{8a}$$

where the matrix $\mathbf{D}_I$ is not necessarily symmetric and may depend upon the current stress state or indeed upon the previous history of the body.

Similarly the arguments remain valid for any incremental form of Darcy's Law

$$\dot{\mathbf{v}} = \mathbf{k}_I\nabla\dot{p}. \tag{8b}$$

These considerations then lead to a set of finite element equations

$$\mathbf{K}_I\dot{\boldsymbol{\delta}} - \mathbf{L}^T\dot{\mathbf{q}} = \dot{\mathbf{m}} \tag{9a}$$

$$-\mathbf{L}\frac{d\dot{\boldsymbol{\delta}}}{dt} - \boldsymbol{\Phi}_I\dot{\mathbf{q}} = \mathbf{0} \tag{9b}$$

where

$$\mathbf{K}_I = \int \mathbf{B}^T\mathbf{D}_I\mathbf{B}\, dV$$

$$\dot{\mathbf{m}} = \int \mathbf{C}^T\dot{\mathbf{T}}\, dS + \int \mathbf{C}^T\dot{\mathbf{F}}\, dV$$

$$\boldsymbol{\Phi}_I = \int \mathbf{E}^T\mathbf{k}_I\mathbf{E}\, dV$$

and the definition of $\mathbf{L}$ remains unchanged.

In this treatment it will be assumed that the matrix of permeabilities $\mathbf{k}_I$ is independent of the previous history of the body although permeability may be anisotropic and vary from point to point. This means that

$$\mathbf{k}_I = \mathbf{k}$$

and thus

$$\boldsymbol{\Phi}_I = \boldsymbol{\Phi}$$

and so eqn (9b) may be integrated to give

$$-\mathbf{L}\frac{d\boldsymbol{\delta}}{dt} - \boldsymbol{\Phi}\mathbf{q} = \mathbf{0}. \tag{9c}$$

The form of the matrix $\mathbf{D}_I$ will be discussed in the next section.

5. INCREMENTAL LAW FOR A PERFECTLY PLASTIC MATERIAL

Suppose that the soil skeleton is a perfect elasto–plastic material with the yield criterion.

$$f(\boldsymbol{\sigma}'_e, \mathbf{x}) \leq 0. \tag{10a}$$

An equality in eqn (10a) indicates a plastic state while an inequality indicates that the material is elastic.

Notice that the quantity $\boldsymbol{\sigma}'_c$ occurring in eqn (10a) is the current effective stress which must be distinguished from the excess effective stress $\boldsymbol{\sigma}'$ introduced previously. The two quantities are of course linked by the equation

$$\boldsymbol{\sigma}'_c = \boldsymbol{\sigma}'_0 + \boldsymbol{\sigma}' \tag{10b}$$

where $\boldsymbol{\sigma}'_0$ is the initial effective stress.

The total strain $\boldsymbol{\epsilon}$ can be thought of as being composed of an elastic component $\boldsymbol{\epsilon}_E$ and a plastic component $\boldsymbol{\epsilon}_P$, as follows

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_E + \boldsymbol{\epsilon}_P \tag{11a}$$

where

$$\boldsymbol{\epsilon}_E = -\mathbf{D}_E^{-1}\boldsymbol{\sigma}' \tag{11b}$$

and $\mathbf{D}_E$ is the matrix of elastic constants defined previously.

Suppose that the soil skeleton has the flow rule

$$\dot{\boldsymbol{\epsilon}}_P = -\lambda\,\mathbf{r} \tag{12a}$$

where $\mathbf{r}$ is a prescribed function of the current effective stress $\boldsymbol{\sigma}'_c$ and possibly of position and where λ is a one-signed multiplier which takes the value zero if the material is in an elastic state. The importance of ensuring that λ has the correct sign has been discussed in Ref. [14].

Equation (12a) is often written in the form

$$\dot{\boldsymbol{\epsilon}}_P = -\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}'} \tag{12b}$$

where g is called the plastic potential. Many previous investigators have limited their attention to the somewhat artificial case of material with an associated flow rule in which the plastic potential g coincides with the yield surface f and the plastic strain rates are directed along the outward normal to the yield surface.

Equations (11a, b) and (12a) can be combined to give

$$\dot{\boldsymbol{\sigma}}' = -\lambda\,\mathbf{D}_E\mathbf{r} - \mathbf{D}_E\dot{\boldsymbol{\epsilon}}. \tag{12c}$$

The multiplier λ can be eliminated from this equation by observing that for a plastic element the stress state must remain on the yield surface and thus

$$0 = \mathbf{s}^T\dot{\boldsymbol{\sigma}}' \tag{12d}$$

where

$$\mathbf{s} = \frac{\partial f}{\partial \boldsymbol{\sigma}'} = \begin{bmatrix} \dfrac{\partial f}{\partial \sigma'_{11}} \\ \vdots \end{bmatrix}.$$

Equation (12d) then implies that

$$\lambda = -\frac{\mathbf{s}^T\mathbf{D}_E\dot{\boldsymbol{\epsilon}}}{\mathbf{s}^T\mathbf{D}_E\mathbf{r}} \tag{12e}$$

and thus substituting into eqn (12c) it is found that

$$\dot{\boldsymbol{\sigma}}' = \mathbf{D}_I\dot{\boldsymbol{\epsilon}} \tag{13a}$$

where

$$\mathbf{D}_I = \mathbf{D}_E - \frac{\boldsymbol{\alpha}\boldsymbol{\beta}^T}{\mathbf{s}^T\boldsymbol{\alpha}} \tag{13b}$$

and

$$\boldsymbol{\alpha} = \mathbf{D}_E \mathbf{r}$$

$$\boldsymbol{\beta} = \mathbf{D}_E \mathbf{s}.$$

Equation (13a) is precisely of the form used in the development of the previous section. Notice that $\mathbf{D}_t$ is just the matrix of elastic constants $\mathbf{D}_E$ diminished by the dyadic matrix $\boldsymbol{\alpha\beta}^T/\{\mathbf{s}^T\boldsymbol{\alpha}\}$ which depends upon the current effective stress $\boldsymbol{\sigma}'_c$. It is also perhaps worth remarking that $\mathbf{D}_t$ unlike $\mathbf{D}_E$ is not in general symmetric† and consequently the incremental stiffness matrix $\mathbf{K}_t$ occurring in eqn (9a) will not in genrral be symmetric.

In a later section two example problems involving plane strain conditions and simple cohesive-frictional soil are considered. For this purpose it is convenient to temporarily drop the tensor and matrix notation of the previous sections and introduce the cartesian axes x, y, z. If it is assumed that the direction of zero strain is the Z direction, the Coulomb failure criterion is, in the usual notation:

$$(\sigma'_x - \sigma'_y)^2 + 4\sigma'^2_{xy} - \sin^2\phi'(\sigma'_x + \sigma'_y + 2c'\cot\phi')^2 \leq 0 \tag{14a}$$

where c', ϕ' are the drained cohesion and angle of friction of the soil skeleton and σ'_x, σ'_y, σ'_{xy} denote the current components of the effective stress tensor (see Fig. 2).

For such a material the vector s has the form

$$\mathbf{s} = -4R\begin{bmatrix}\cos 2\mu - \cos 2\theta \\ \cos 2\mu + \cos 2\theta \\ -2\sin 2\theta\end{bmatrix} \tag{14b}$$

where $\mu = \pi/4 - \phi'/2$,
R is the radius of the Mohr circle of stress and
θ is the angle between the principal stress direction and the x-axis.

If this material had an associated flow rule then the strain rates would be given by

$$\begin{bmatrix}\dot{\epsilon}_x \\ \dot{\epsilon}_y \\ \dot{\gamma}_{xy}\end{bmatrix} = 4\lambda R\begin{bmatrix}\cos 2\mu - \cos 2\theta \\ \cos 2\mu + \cos 2\theta \\ -2\sin 2\theta\end{bmatrix}. \tag{14c}$$

Such a material would dilate at a rate far greater than observed in real soils, this observation led Davis[9] with Hansen[25] to propose a class of non-associated flow rules of the form

$$\begin{bmatrix}\dot{\epsilon}_x \\ \dot{\epsilon}_y \\ \dot{\gamma}_{xy}\end{bmatrix} = 4\lambda R\begin{bmatrix}\cos 2\nu - \cos 2\theta \\ \cos 2\nu + \cos 2\theta \\ -2\sin 2\theta\end{bmatrix} = -\lambda\mathbf{r} \tag{14d}$$

where $\nu = \pi/4 - \psi/2$ and ψ is an angle which plays the same role in relation to the plastic strain rates as ϕ' does for the stresses. A material with an associated flow rule would be characterised by $\psi = \phi'$ while a material which deformed at constant volume would have $\psi = 0$.

6. DRAINED AND UNDRAINED BEHAVIOUR

The drained and undrained behaviour represent two important aspects of the behaviour of a saturated clay.

Consider first the drained behaviour. This is relatively straight-forward, for if loads are applied sufficiently slowly, so that no excess pore pressures develop, the material behaves as a perfectly plastic solid with a yield criterion given by eqn (10a) and a flow rule given by eqn (12a).

The undrained behaviour is more complex, if loads are applied sufficiently quickly or

†Symmetry only occurs in the case of an associated flow rule.

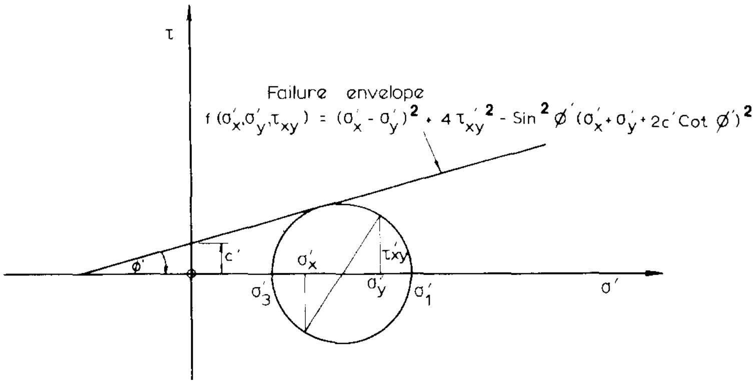

Fig. 2. The Mohr–Coulomb failure criterion.

equivalently if all drainage is prevented, reasoning similar to that used to obtain eqn (12e) shows that there can be no volume change.

Suppose that a given element is loaded under undrained conditions so that it passes through a series of elastic states until it eventually reaches a plastic state.

The stress–strain law for such an element is

$$\boldsymbol{\sigma}'_c = \boldsymbol{\sigma}'_0 - \mathbf{D}_E\boldsymbol{\epsilon} \tag{15a}$$

where $\boldsymbol{\sigma}'_c$ is the current stress state and $\boldsymbol{\sigma}'_0$ is the initial stress state.

Because the loading occurs under undrained conditions there can be no volume change and thus

$$\mathbf{e}^T\boldsymbol{\epsilon} = 0$$

where

$$\mathbf{e}^T = (1, 1, 1, 0, 0, 0). \tag{15b}$$

This means that

$$0 = \mathbf{h}^T(\boldsymbol{\sigma}'_c - \boldsymbol{\sigma}'_0) \tag{15c}$$

where

$$\mathbf{h} = -\mathbf{D}_E^{-1}\mathbf{e}$$

so that $\mathbf{h}$ represents the elastic strain induced by unit hydrostatic stress.†

Equation (15c) can be interpreted as the equation of a plane passing through $\boldsymbol{\sigma}'_0$ and normal to the vector $\mathbf{h}$. Thus first yield will occur on the intersection of the plane (15c) and yield surface (10a). This is shown schematically in Fig. 3. Equation (15c) can be re-written as

$$0 = \mathbf{h}^T(\boldsymbol{\sigma} - p\,\mathbf{e})$$

so that the excess pore pressure at failure will be

$$p = \frac{\mathbf{h}^T\boldsymbol{\sigma}}{\mathbf{h}^T\mathbf{e}}. \tag{15d}$$

The question now arises as to whether the element will behave as a perfect plastic material or whether it will exhibit strain hardening or softening.

If the material is to behave as a perfectly plastic material then it will continue to deform indefinitely under constant stresses. Under such conditions

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_p \tag{16a}$$

†For a material with an isotropic elastic skeleton the vector $\mathbf{h}$ is parallel to the vector $\mathbf{e}$.

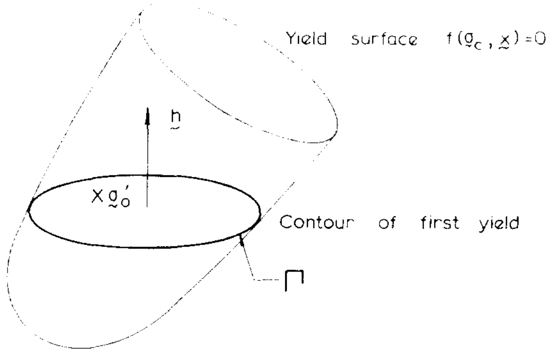

Fig. 3.

because

$$\dot{\boldsymbol{\epsilon}}_E = \mathbf{D}_E^{-1}\dot{\boldsymbol{\sigma}}'$$

and $\dot{\boldsymbol{\sigma}}' = \mathbf{0}$, since it was assumed that the stresses were held constant.

The element is still in the undrained state and thus

$$\mathbf{e}^T\dot{\boldsymbol{\epsilon}} = \mathbf{e}^T\dot{\boldsymbol{\epsilon}}_p = 0 \tag{16b}$$

and therefore

$$\mathbf{e}^T\mathbf{r} = 0. \tag{16c}$$

Thus an element will only behave in a perfectly plastic manner if its flow rule involves zero plastic volume change. For this reason it is often convenient to assume that the material obeys such a flow rule. Many clays are not strongly dilatent and in any case all soils must ultimately deform at constant volume. The assumption of zero plastic volume change therefore provides quite a reasonable approximation to the true behaviour of the clay.

Assuming that the element does behave in a perfectly plastic manner, i.e. there is no plastic volume change it is interesting to interpret the undrained failure criterion shown in Fig. 3 in terms of total stresses. Clearly failure can only occur when the effective stress lies on the curve Γ shown in Fig. 3.

Recalling that

$$\boldsymbol{\sigma}'_c = \boldsymbol{\sigma}_c - p\,\mathbf{e}$$

it can be seen that failure will occur whenever the total stress lies on the cylindrical surface passing through Γ and with its generators parallel to **e**. It should be noted that this yield surface depends upon the initial effective stresses σ'_0 and thus a body with a non-uniform initial effective stress distribution will have an undrained strength behaviour that varies from point to point. It is also interesting to note that a skeleton with anisotropic elastic properties but with an isotropic yield condition will have an isotropic yield condition for drained failure but an anisotropic yield condition for undrained failure.

Materials having flow rules which involve non-zero plastic volume changes will not in general exhibit a perfect plastic undrained behaviour but will strain harden. This is clearly illustrated by the simple example shown schematically in Fig. 4. In this figure ABCD is the typical cross-section of an element of isotropic soil which has a Mohr–Coulomb failure criterion given by eqn (14a) and a non-associated flow rule given by eqn (14d), and which is being tested under conditions of plane strain. Initially it will be assumed that the element is loaded by major and minor principal stresses σ'_{10}, σ'_{30} and that subsequently that these are increased to $\sigma'_{10}+\sigma'_1$, $\sigma'_{30}+\sigma'_3$ under undrained conditions.

It is easily shown that while the material remains elastic the principal strains ϵ_1, ϵ_3 are given by

$$\epsilon_1 = -\frac{1}{E_1}(\sigma'_1 - \nu_1\sigma'_3) \tag{17a}$$

$$\epsilon_3 = -\frac{1}{E_1}(\sigma_3' - \nu_1\sigma_1') \tag{17b}$$

where

$$E_1 = \frac{E'}{1-\nu'^2} \quad \nu_1 = \frac{\nu'}{1-\nu'}$$

and E', ν' are the Young's modulus and Poisson's ratio of the soil skeleton.

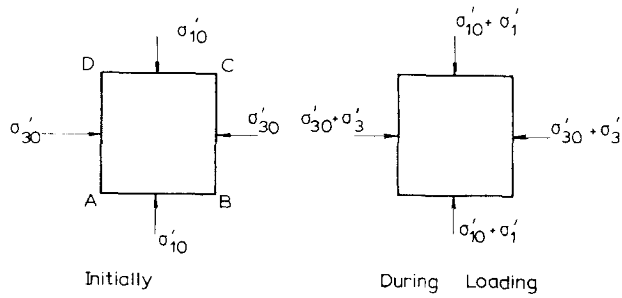

Fig. 4.

The condition of zero volume strain implies that $\epsilon_1 = -\epsilon_3$ and that $\sigma_1' = -\sigma_3'$. Plastic failure will first occur when

$$\frac{\sigma_{10}' + \sigma_1' + c'\cot\phi'}{\sigma_{30}' + \sigma_3' + c'\cot\phi'} = N_\phi = \frac{1+\sin\phi'}{1-\sin\phi'} \tag{18a}$$

that is when

$$\sigma_1 - \sigma_3 = \sigma_1' - \sigma_3' = 2c_{uI} \tag{18b}$$

where c_{uI} is the initial undrained shear strength given by

$$c_{uI} = \frac{N_\phi(\sigma_{30}' + c'\cot\phi') - (\sigma_{10}' + c'\cot\phi')}{1+N_\phi}. \tag{18c}$$

The corresponding strain would then be given by

$$\epsilon_1 - \epsilon_3 = -\frac{1}{2G'}(\sigma_1 - \sigma_3)$$

$$= -\frac{c_{uI}}{G'} \tag{18d}$$

where $G' = E'/2(1+\nu')$ is the shear modulus of the soil skeleton.

Suppose now, that over a period of time Δt, the element is given an additional effective stress increment $\dot{\sigma}_1'\Delta t$, $\dot{\sigma}_3'\Delta t$. If the element remains plastic then it follows from eqns (14a) and (14d) that

$$\frac{\dot{\sigma}_1'}{\dot{\sigma}_3'} = N_\phi \tag{19a}$$

$$\frac{\dot{\epsilon}_{1_p}}{\dot{\epsilon}_{3_p}} = -\frac{1}{N_\psi} = -\frac{(1-\sin\psi)}{(1+\sin\psi)} \tag{19b}$$

where the subscript p denotes the plastic portion of strain. Now the total strain rates are given by

$$\dot{\epsilon}_1 = -\frac{1}{E_1}(\dot{\sigma}_1' - \nu_1\dot{\sigma}_3') + \dot{\epsilon}_{1_p}$$

$$\dot{\epsilon}_3 = -\frac{1}{E_1}(\dot{\sigma}_3' - \nu_1\dot{\sigma}_1') + \dot{\epsilon}_{3_p}$$

and thus using eqns (19a, b) it is found that

$$\frac{\dot{\sigma}_1' - \dot{\sigma}_3'}{2G'} = \frac{\dot{\sigma}_1 - \dot{\sigma}_3}{2G'} = -M\{\dot{\epsilon}_1 - \dot{\epsilon}_3\} \tag{20}$$

where

$$M = \frac{(N_\psi - 1)(N_\phi - 1)(1 + \nu_1)}{2\{(N_\phi - \nu_1)N_\psi + (1 - \nu_1 N_\phi)\}}.$$

Equations (18c) and (20) can be combined to give the complete stress strain curve, this is shown in Fig. 5. It is evident that there will be a strain hardening type of behaviour except when $\psi = 0$.

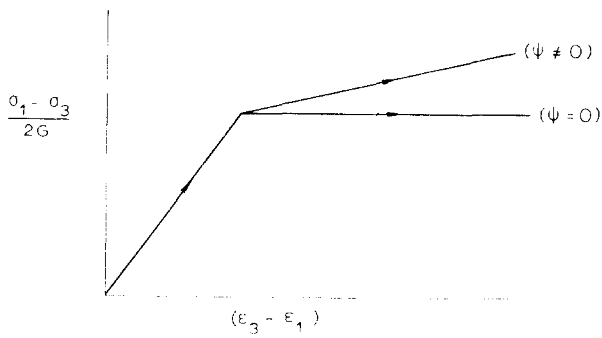

Fig. 5.

7. NUMERICAL INTEGRATION OF THE FINITE ELEMENT EQUATIONS

Consider first the case of a consolidating soil with an elastic skeleton then

$$\mathbf{K}_E\boldsymbol{\delta} - \mathbf{L}^T\mathbf{q} = \mathbf{m} \tag{21a}$$

$$-\mathbf{L}\frac{d\boldsymbol{\delta}}{dt} - \boldsymbol{\Phi}\mathbf{q} = \mathbf{0}. \tag{21b}$$

Suppose that the solution $(\boldsymbol{\delta}_1, \mathbf{q}_1)$ is known at time t_1 and it is required to evaluate the solution $(\boldsymbol{\delta}_2, \mathbf{q}_2)$ at time $t_2 = t_1 + \Delta t$. Equation (21b) can be integrated approximately in the form

$$-\mathbf{L}\{\delta_2 - \delta_1\} - \Phi\{\alpha\mathbf{q}_2 + (1 - \alpha)\mathbf{q}_1\}\Delta t = \mathbf{0} \tag{21c}$$

where α defines the particular integration rule used, for example $\alpha = \frac{1}{2}$ corresponds to the trapezoidal rule.

Equations (21a, b) may now be written in the form

$$\begin{bmatrix} \mathbf{K}_E & -\mathbf{L}^T \\ -\mathbf{L} & -\alpha\Delta t\boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_2 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{m}_2 \\ -\mathbf{L}\boldsymbol{\delta}_1 + (1 - \alpha)\Delta t\boldsymbol{\Phi}\mathbf{q}_1 \end{bmatrix}. \tag{21d}$$

Thus, if the solution is known at t_1, it can be found at t_2 and so the solution can be marched forward in time in the usual way. The stability of this integration scheme has been examined in [24] and it was shown that the process is unconditionally stable provided $\alpha \geq \frac{1}{2}$.

The case when the soil skeleton obeys a general incremental law may be treated in a very similar fashion. In this case eqn (21a) is replaced by the equation

$$\mathbf{K}_t\dot{\boldsymbol{\delta}} - \mathbf{L}^T\dot{\mathbf{q}} = \dot{\mathbf{m}}.$$

Now integrating this equation with respect to time it is found that to sufficient approximation:

$$\bar{\mathbf{K}}_t\{\boldsymbol{\delta}_2 - \boldsymbol{\delta}_1\} - \mathbf{L}^T\{\mathbf{q}_2 - \mathbf{q}_1\} = \{\mathbf{m}_2 - \mathbf{m}_1\}$$

where $\bar{\mathbf{K}}_I$ represents an average of $\mathbf{K}_I$ over the interval (t_1, t_2). There are of course many ways of calculating this average, one such method would be to set

$$\bar{\mathbf{K}}_I = \alpha \bar{\mathbf{K}}_{I1} + (1-\alpha)\bar{\mathbf{K}}_{I2}$$

where as before the subscript 1, 2 denote the values at t_1, t_2 respectively. A particularly convenient form is to take $\bar{\mathbf{K}}_I$ to be the value of $\mathbf{K}_I$ for $t = \frac{1}{2}(t_1+t_2)$, $\boldsymbol{\delta} = \frac{1}{2}(\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2)$, $\mathbf{q} = \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2)$ etc.

The approximating equations are therefore

$$\begin{bmatrix} \bar{\mathbf{K}}_I, & -\mathbf{L}^T \\ -\mathbf{L}, & -\alpha\Delta t\boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_2 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{K}}_I\boldsymbol{\delta}_1 - \mathbf{L}^T\mathbf{q}_1 + \{\mathbf{m}_2 - \mathbf{m}_1\} \\ -\mathbf{L}\boldsymbol{\delta}_1 + (1-\alpha)\Delta t\boldsymbol{\Phi}\mathbf{q}_1 \end{bmatrix}. \tag{22a}$$

Equations (22a) are a set of non-symmetric non-linear equations, a convenient method of solving these equations is to use an adaptation of the initial stress approach developed in [13]. Equations (22a) are then solved iteratively using the following algorithm.

$$\begin{bmatrix} \mathbf{K}_R, & -\mathbf{L}^T \\ -\mathbf{L}, & -\alpha\Delta t\boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_2^{(n+1)} \\ \mathbf{q}_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{(n)} \\ \mathbf{b} \end{bmatrix} \tag{22b}$$

where

$$\mathbf{a}^{(n)} = \mathbf{K}_R\boldsymbol{\delta}_2^{(n)} - \bar{\mathbf{K}}_I^{(n)}(\boldsymbol{\delta}_2^{(n)} - \boldsymbol{\delta}_1) - \mathbf{L}^T\mathbf{q}_1 + \{\mathbf{m}_2 - \mathbf{m}_1\}$$

$$\mathbf{b} = -\mathbf{L}\boldsymbol{\delta}_1 + (1-\alpha)\Delta t\boldsymbol{\Phi}\mathbf{q}_1$$

and $\mathbf{K}_R$ is some symmetric reference stiffness matrix, for elasto-plastic calculations it is convenient to take $\mathbf{K}_R = \mathbf{K}_E$ the elastic stiffness matrix.

Notice that if $\alpha\Delta t$ is kept constant, eqn (22b) can be viewed as resulting from a given "structure" with several "load sets" and so it is only necessary to perform one Crout–Cholesky decomposition of the matrix.

$$\begin{bmatrix} \mathbf{K}_R & -\mathbf{L}^T \\ -\mathbf{L} & -\alpha\Delta t\boldsymbol{\Phi} \end{bmatrix}$$

which may be used in the iterative solution of eqn (22b) and in the subsequent iterative solutions which are necessary to find the values at $t_3 = t_2 + \Delta t$, $t_4 = t_3 + \Delta t, \ldots$ etc.

In order to start the solution process it is necessary to know the solution at some instant of time. This can be done either by performing a separate elastic, or if necessary elasto–plastic undrained analysis, using undrained parameters or alternatively by adopting the artifice that the loads increase from the initial value of zero over a short period of time.

8. ILLUSTRATIVE EXAMPLES

There do not seem to be any published rigorous solutions to problems which involve both consolidation and yielding. However it is possible to provide quite a searching test, to the correctness of the theory developed in the previous section, by examining its ability to predict the drained and undrained behaviour which correspond to infinitely slow and infinitely fast loading rates respectively.

In this section attention will be restricted to the plane strain of an isotropic saturated soil with a perfectly elasto–plastic skeleton which obeys the Mohr–Coulomb failure criterion, eqn (14a) and has a non-associated flow rule (14d) and which is initially in a state of zero effective stress. For definiteness it will be assumed that the soil parameters have the following values

$$E'/c' = 200.0$$

$$\nu' = 0 \quad \text{or } 0.3$$

$$\phi' = 30^{\circ},$$

$$\psi = 0.$$

It will be convenient to introduce the dimensionless time T_v:

$$T_v = c_v t / a^2,$$

where

$$c_v = \frac{k}{\gamma_w} \frac{(1-\nu')}{(1-2\nu')} \frac{E'}{(1+\nu')} = \text{the coefficient of one dimensional consolidation,}$$

and

a = some reference length which is taken to be the inner radius of the cylinder in the first example and the breadth of the footing in the second.

It is also convenient to define a load rate parameter

$$\omega = \frac{d(p/c')}{d(T_v)},$$

where p is the internal pressure acting on the cylinder in the first example and the average pressure acting on the footing in the second.

The undrained behaviour of this material will be governed by the undrained parameters

$$E_u = \frac{3E'}{2(1+\nu')},$$

$$\nu_u = 0.5$$

$$c_u = \frac{2c'\sqrt{N_\phi}}{1+N_\phi}, \quad \text{(because of the assumed initial conditions)}$$

and

$$\phi_u = 0.$$

As a first example consider the thick-walled cylinder shown in Fig. 6a, b. The cylinder is subjected to an internal pressure p and zero external pressure. For numerical calculations it was assumed that $b/a = 2$, and $\nu' = 0.0$.

In order to approximate an undrained behaviour the cylinder was first loaded at the "fast" rate of $\omega = 9$. In Fig. 7 the results of the elasto–plastic consolidation finite element analysis are compared with those from a straight elasto–plastic finite element analysis and the curve from the analytic elasto–plastic solution[26], both of which take no account of consolidation and use the undrained strength and deformation parameters. The three analyses can be seen to agree quite closely, although the results for the elasto–plastic consolidation analysis are a little below the other two, suggesting that a rate of $\omega = 9$ is not fast enough to fully produce undrained conditions. However a substantial increase in ω would reduce the accuracy of the elasto–plastic consolidation analysis due to the difficulty of representing very high pore pressure gradients unless a much finer finite element mesh were used adjacent to drainage boundaries.

The drained behaviour was approximated by loading at a rate one hundredth of that used to approximate the undrained case, viz $\omega = 0{\cdot}09$. This was again compared with a straight

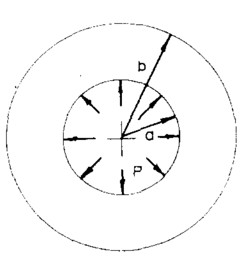

Fig. 6a.

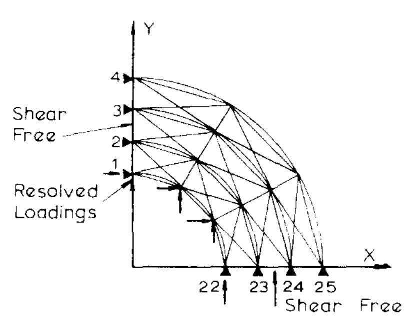

Fig. 6b. Finite element mesh used.

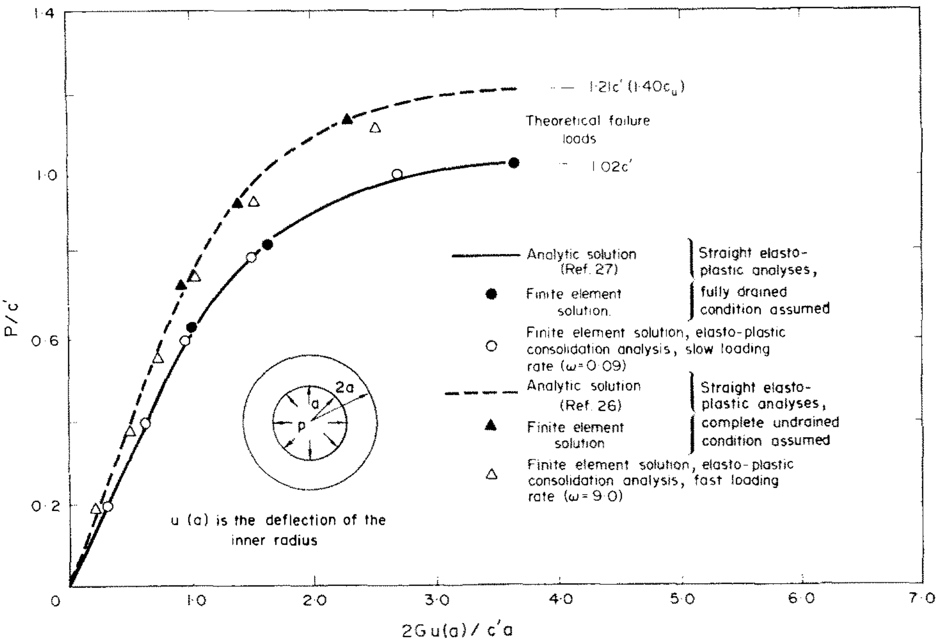

Fig. 7.

elasto-plastic analysis and an analytic solution (27) both of which use drained parameters and take no account of consolidation. The three analyses may be seen to be in close agreement.

A second example is shown schematically in Fig. 8. In this figure A'OA is a typical section of a smooth, perfectly flexible, uniformly loaded, strip footing acting on a layer of soil resting on a smooth rigid base. In order to completely define the problem it is assumed that there is no resultant horizontal force on any vertical section. For the numerical calculation it was assumed that $h/a = 2$ and $\nu' = 0.3$.

This problem was solved for a whole succession of constant load rates varying from $\omega = 0.143$ to $\omega = 143.0$. The two extreme cases were compared with the results of elasto-plastic finite element analyses using drained and undrained parameters and again show close agreement. It can be seen that the undrained solution assymptotes to the analytic collapse load for $\phi = 0$[28], while the drained solution assymptotes to the analytic collapse load for a material with $\phi = \phi'$ and $\psi = \phi'$, [28], (see Fig. 9).

In Fig. 10 the failure load is plotted against the rate of loading and it may be observed that as the loading rate is increased there is a smooth transition from drained failure to undrained failure, and that for loading rates less than about $\omega = 0.143$ the soil behaves in a drained manner while for loading rates greater than $\omega = 143.0$ the soil has an undrained behaviour.

Pore pressure distributions along the centre line of the footing are shown for a fast loading case $\omega = 14.3$ (Fig. 11a) and for a slow rate $\omega = 0.143$ (Fig. 11b). The dotted contours of excess pore pressure are those corresponding to the values after plastic failure has occurred.

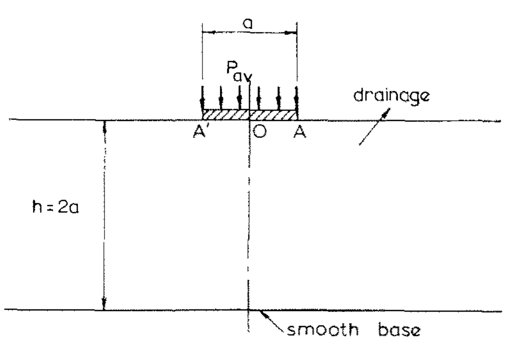

Fig. 8a. Footing on finite layer.

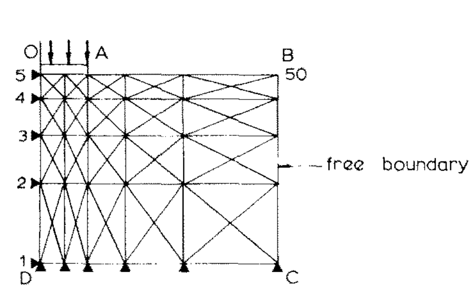

Fig. 8b. Finite element mesh used for footing.

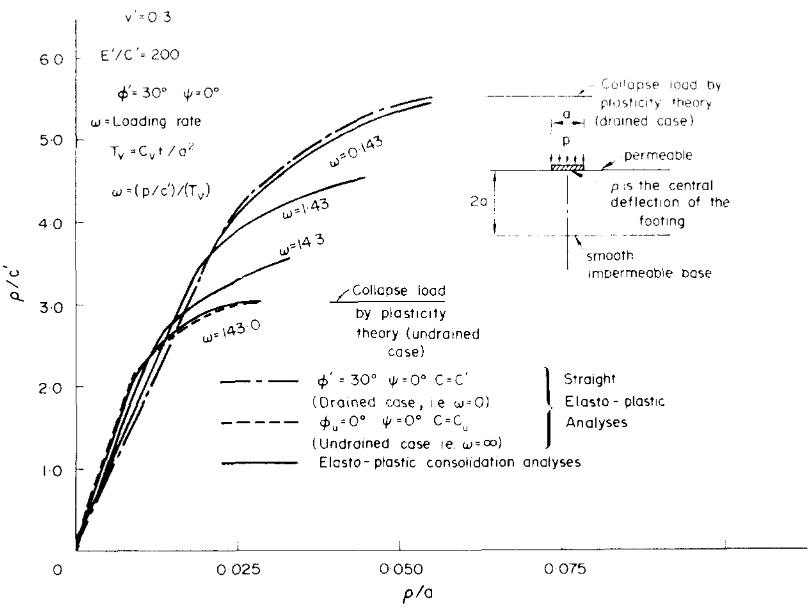

Fig. 9.

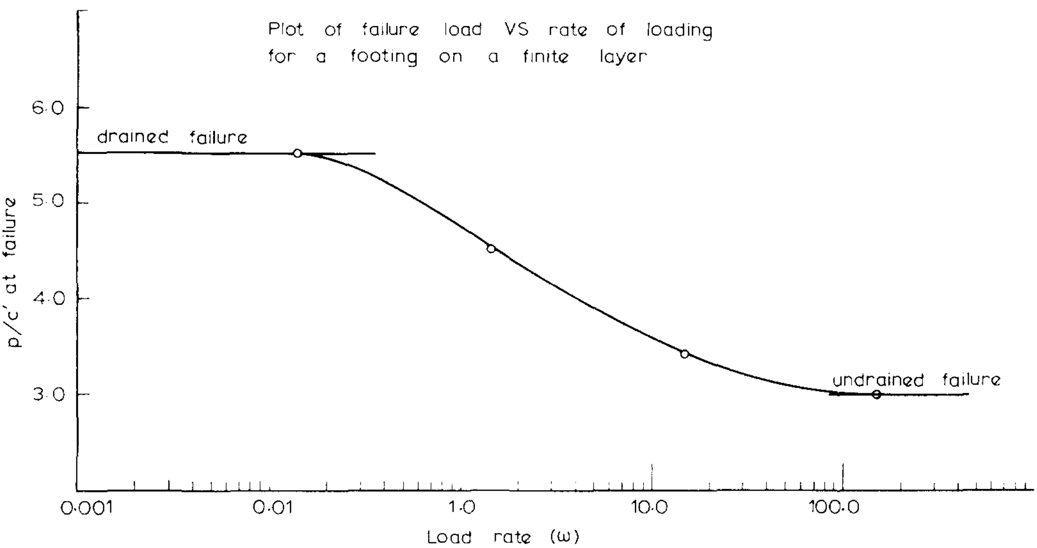

Fig. 10.

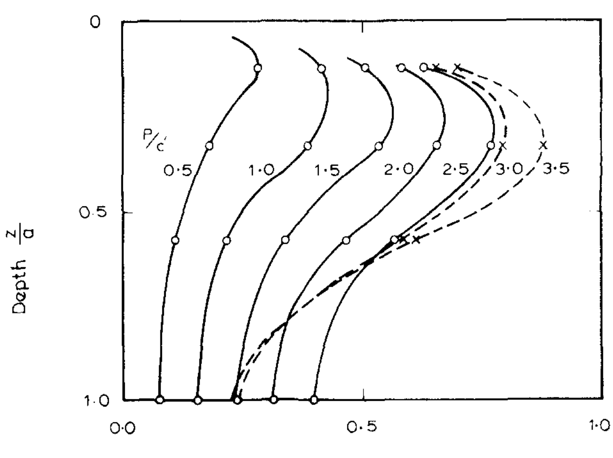

Fig. 11a. Fast load rate ω = 14.3.

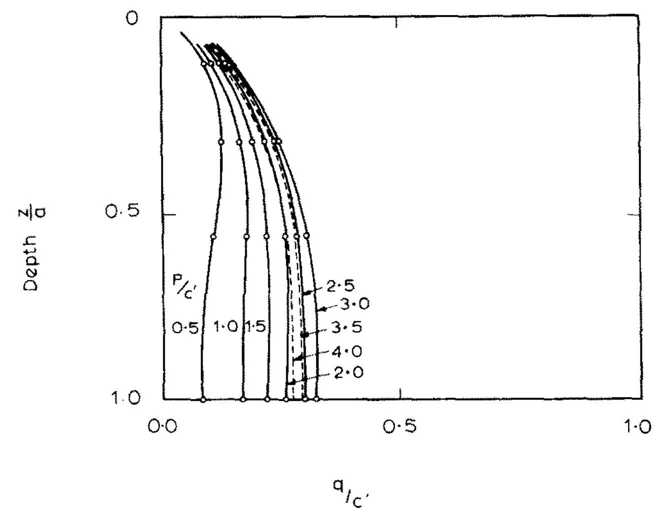

Fig. 11b. Slow load rate ω = 1.43.

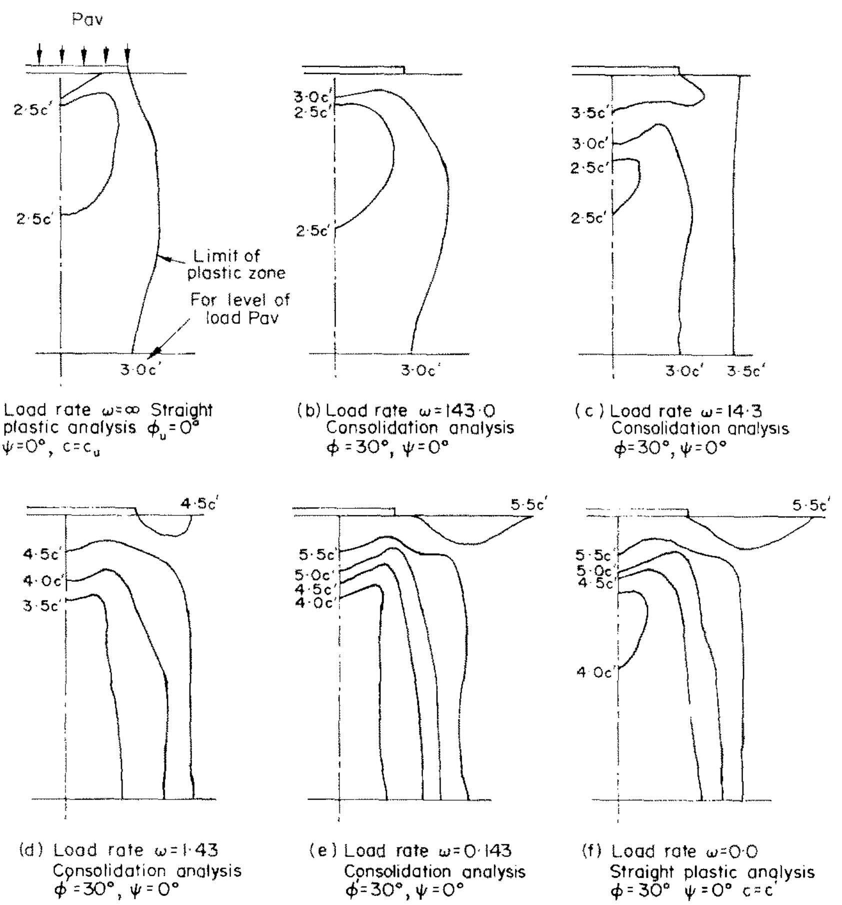

Fig. 12.

Finally, Figs. 12a–12f show the growth of plastic regions in the material beneath the footing. Figures 12a and 12f are the results obtained from an elasto–plastic program and represent the two extremes of drained and undrained failures. Figures 12b–12e are failure contours for progressively slower loading rates, so that comparisons can be made between Figs. 12a, 12b and Figs. 12e, 12f.

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REFERENCES

1. R. Hill, *The Mathematical Theory of Plasticity*. Clarendon Press, Oxford (1950).
2. D. C. Drucker, H. J. Greenberg and W. Prager, *J. Appl. Mech.* **18**, 371 (1951).
3. D. C. Drucker, W. Prager and H. J. Greenberg, *Quart J. Mech. App. Math.* **9**, (1952).
4. D. C. Drucker, *J. Mech. Phys. Solids* **1**, 217 (1953).
5. R. T. Shield, *Quart. Appl. Math.* **11**, 61 (1953).
6. R. T. Shield, *J. Math. Phys.* **33**, 144 (1954).
7. R. T. Shield, *J. Mech. Phys. Solids* **4**, 10 (1955).
8. V. V. Sokolovsky, *Statics of Soil Media*. Butterworths, London (1956).
9. E. H. Davis, *Soil Mechanics—Selected Topics* (Edited by I. K. Lee), Chap. 6. Butterworths, London (1968).
10. J. H. Argyris, *J. R. Aeronaut. Soc.* **69**, 633 (1965).
11. P. V. Marcal and I. P. King, *Int. J. Mech. Sci.* **9**, 143 (1967).
12. Y. Yamanada, N. Yoshimura and T. Sakurai, *Int. J. Mech. Sci.* **10**, 343 (1968).
13. O. C. Zienkiewicz, S. Valliappan and I. P. King, *Int. J. Num. Meth. Engng* **1**, 75 (1969).
14. E. H. Davis and J. R. Booker, *Proc. Symp. Role of Plasticity in Soil Mechanics*, Cambridge (1973).
15. E. H. Davis and H. G. Poulos, *Geotechnique* **22**, 95 (1972).
16. M. A. Biot, *J. App. Phys.* **12**, 155 (1941).
17. M. A. Biot, *J. App. Phys.* **12**, 426 (1941).
18. M. A. Biot, *J. App. Phys.* **27**, 459 (1956).
19. M. A. Biot, *J. App. Mech.* **23**, 91 (1956).
20. R. S. Sandhu and E. L. Wilson, *Proc. Amer. Soc. Civ. Engrs* **95**, SM3, 641 (1969).
21. J. T. Christian and J. W. Boehmer, *Proc. Amer. Soc. Civ. Engrs* **96**, SM4, 1435 (1970).
22. C. T. Hwang, N. R. Morgenstein and D. W. Murray, *Can. Geotechnical J.* **8**, 109 (1971).
23. J. R. Booker, *Quart. J. Mech. App. Math.* **26**, 457 (1973).
24. J. R. Booker and J. C. Small, *Int. J. Solids Structures* **11**, 907 (1975).
25. B. Hansen, *Proc. Conf. Earth Pressure Probl. Brussels*, 39 (1958).
26. W. Prager and P. G. Hodge, *Theory of Perfectly Plastic Solids*, p. 95. Wiley, NewYork (1951).
27. C. Chapman, Private communication.
28. J. R. Booker, *Quart. J. Mech. App. Math.* **25**, 63 (1972).